用 Rosenbluth 势表示的 Fokker-Planck 碰撞算子

从碰撞项的定义出发推导出 Rosenbluth 势表示的 Fokker-Planck 碰撞算子。Rosenbluth 势表示的 Fokker-Planck 碰撞算子将库仑碰撞，微分散射截面，碰撞的转移几率联系起来。

其基本思路为

1. 根据碰撞项的定义

$$(\frac{\delta f_a}{\delta t})_c = \frac{f_a(\mathbf{r}, \mathbf{v}_a, t) - f_a(\mathbf{r}, \mathbf{v}_a, t - \Delta t)}{\Delta t} \tag{1}$$

其中 $f_a(\mathbf{r}, \mathbf{v}_a, t)$ 为 t 时刻，粒子处在位置 $\mathbf{r}$，速度 $\mathbf{v}_a$ 的概率分布密度，只要求出 $f_a(\mathbf{r}, \mathbf{v}_a, t)$ 和 $f_a(\mathbf{r}, \mathbf{v}_a, t - \Delta t)$ 的表达式，即可求出碰撞项 $(\frac{\delta f_a}{\delta t})_c$。

2. 假定每一次碰撞使速度粒子速度发生一个变化 $\mathbf{v}_a \to \mathbf{v}_a + \Delta\mathbf{v}_a$，则有

$$\begin{aligned} f_a(\mathbf{r}, \mathbf{v}_a, t) = \int & d(\Delta\mathbf{v}_a) f_a(\mathbf{r}, \mathbf{v}_a - \Delta\mathbf{v}_a, t - \Delta t) \\ & \cdot \psi(\mathbf{v}_a - \Delta\mathbf{v}_a, \Delta\mathbf{v}_a, t - \Delta t) \end{aligned} \tag{2}$$

其中 $\psi(\mathbf{v}_a - \Delta\mathbf{v}_a, \Delta\mathbf{v}_a, t - \Delta t)$ 为转移概率分布密度，是 $t - \Delta t$ 时刻速度 $\mathbf{v}_a - \Delta\mathbf{v}_a$，粒子速度改变量 $\Delta\mathbf{v}_a$ 的函数，表示粒子在 t 时刻处于位置 $\mathbf{r}$，速度 $\mathbf{v}$ 的概率分布密度 $f_a(\mathbf{r}, \mathbf{v}_a, t)$，为粒子在 $t - \Delta t$ 时刻处在位置 $\mathbf{r}$，速度为 $\mathbf{v}_a - \Delta\mathbf{v}_a$ 的概率密度 $f_a(\mathbf{r}, \mathbf{v}_a - \Delta\mathbf{v}_a, t - \Delta t)$ 在 Δt 时间间隔内以转移几率 $d(\Delta\mathbf{v}_a) \cdot \psi(\mathbf{v}_a - \Delta\mathbf{v}_a, \Delta\mathbf{v}_a, t - \Delta t)$ 转移而得。

3. 两个假设

(1). 转移概率 ψ 与时间无关，即 $\psi = \psi(\mathbf{v}_a - \Delta\mathbf{v}_a, \Delta\mathbf{v}_a)$，并且有归一化条件

$$\int \psi(\mathbf{v}_a - \Delta\mathbf{v}_a, \Delta\mathbf{v}_a) d(\Delta\mathbf{v}_a) = 1 \tag{3}$$

这个为马尔科夫近似。意味着碰撞使状态发生变化的概率和历史无关。

(2). 每次碰撞引起的粒子速度改变 $\Delta\mathbf{v}_a$ 足够小，可以用它来作为小量来展开 f_a 和 ψ。

4. 将方程 (2) 中 $f_a(\mathbf{r}, \mathbf{v} - \Delta\mathbf{v}, t - \Delta t)$ 在 $f_a(\mathbf{r}, \mathbf{v}, t - \Delta t)$ 处按 $\Delta\mathbf{v}$ 作泰

勒展开到二阶并化简得 Fokker-Planck 方程

$$\begin{aligned}
f_a(\mathbf{r},\mathbf{v}_a,t) = &\int d(\Delta\mathbf{v}_a)[f_a(\mathbf{r},\mathbf{v}_a-\Delta\mathbf{v}_a,t-\Delta t)+(-\Delta\mathbf{v}_a)\cdot\frac{\partial f_a(\mathbf{r},\mathbf{v}_a-\Delta\mathbf{v}_a,t-\Delta t)}{\partial\mathbf{v}_a}]|_{\Delta\mathbf{v}_a=0}\\
&+\frac{1}{2}(\Delta\mathbf{v}_a\Delta\mathbf{v}_a):\frac{\partial^2 f_a(\mathbf{r},\mathbf{v}_a-\Delta\mathbf{v}_a,t-\Delta t)}{\partial\mathbf{v}_a\partial\mathbf{v}_a}|_{\Delta\mathbf{v}_a=0}+\dots]\\
&\times[\psi(\mathbf{v}_a,\Delta\mathbf{v}_a)+(-\Delta\mathbf{v}_a)\cdot\frac{\partial\psi(\mathbf{v}_a-\Delta\mathbf{v}_a,\Delta\mathbf{v}_a)}{\partial\mathbf{v}_a}|_{\Delta\mathbf{v}_a=0}\\
&+\frac{1}{2}(\Delta\mathbf{v}_a\Delta\mathbf{v}_a):\frac{\partial^2\psi(\mathbf{v}_a-\Delta\mathbf{v}_a,\Delta\mathbf{v}_a)}{\partial\mathbf{v}_a\partial\mathbf{v}_a}|_{\Delta\mathbf{v}_a=0}+\dots]\\
=&\int d(\Delta\mathbf{v}_a)[f_a(\mathbf{r}_a,\mathbf{v}_a,t-\Delta t)\psi(\mathbf{v}_a,\Delta\mathbf{v}_a)\\
&-\frac{\partial(f_a(\mathbf{r},\mathbf{v}_a-\Delta\mathbf{v}_a,t-\Delta t)\psi(\mathbf{v}_a-\Delta\mathbf{v}_a,\Delta\mathbf{v}_a))}{\partial\mathbf{v}_a}|_{\Delta\mathbf{v}_a=0}\cdot\Delta\mathbf{v}_a\\
&+\frac{1}{2}\frac{\partial^2(f_a(\mathbf{r},\mathbf{v}_a-\Delta\mathbf{v}_a,t-\Delta t)\psi(\mathbf{v}_a-\Delta\mathbf{v}_a,\Delta\mathbf{v}_a))}{\partial\mathbf{v}_a\partial\mathbf{v}_a}|_{\Delta\mathbf{v}_a=0}:(\Delta\mathbf{v}_a\Delta\mathbf{v}_a)]
\end{aligned}$$

利用转移概率 ψ 的归一化条件，并且 $f_a(\mathbf{r},\mathbf{v}_a,t-\Delta t)$ 与积分无关，则有

$$\begin{aligned}
f_a(\mathbf{r},\mathbf{v}_a,t) =&f_a(\mathbf{r},\mathbf{v}_a,t-\Delta t)-\frac{\partial}{\partial\mathbf{v}}\cdot[f_a(\mathbf{r},\mathbf{v}_a,t-\Delta t)\int d(\Delta\mathbf{v}_a)\psi(\mathbf{v}_a-\Delta\mathbf{v}_a,\Delta\mathbf{v}_a)\Delta\mathbf{v}_a]\\
&+\frac{1}{2}\frac{\partial^2}{\partial\mathbf{v}_a\partial\mathbf{v}_a}:[f_a(\mathbf{r},\mathbf{v}_a,t-\Delta t)\int d(\Delta\mathbf{v}_a)\psi(\mathbf{v}_a-\Delta\mathbf{v}_a,\Delta\mathbf{v}_a)\Delta\mathbf{v}_a\Delta\mathbf{v}_a]
\end{aligned}\tag{4}$$

令

$$\langle\Delta\mathbf{v}_a\rangle=\frac{1}{\Delta t}\int d(\Delta\mathbf{v}_a)\psi(\mathbf{v}_a-\Delta\mathbf{v}_a,\Delta\mathbf{v}_a)\Delta\mathbf{v}_a\tag{5}$$

$$\langle\Delta\mathbf{v}_a\Delta\mathbf{v}_a\rangle=\frac{1}{\Delta t}\int d(\Delta\mathbf{v}_a)\psi(\mathbf{v}_a-\Delta\mathbf{v}_a,\Delta\mathbf{v}_a)\Delta\mathbf{v}_a\Delta\mathbf{v}_a\tag{6}$$

$\langle\Delta\mathbf{v}_a\rangle$ 为动力学摩擦系数，碰撞使粒子系速度减慢，$\langle\Delta\mathbf{v}_a\Delta\mathbf{v}_a\rangle$ 为扩散系数，碰撞使速度分布函数在速度空间弥散，则有

$$\begin{aligned}
(\frac{\delta f_a}{\delta t})_c=&-\frac{\partial}{\partial\mathbf{v}_a}\cdot(f_a(\mathbf{r},\mathbf{v}_a,t-\Delta t)\langle\Delta\mathbf{v}_a\rangle)\\
&+\frac{1}{2}\frac{\partial^2}{\partial\mathbf{v}_a\partial\mathbf{v}_a}:(f_a(\mathbf{r},\mathbf{v}_a,t-\Delta t)\langle\Delta\mathbf{v}_a\Delta\mathbf{v}_a\rangle)
\end{aligned}\tag{7}$$

将方程 (7) 带入玻尔兹曼方程

$$\frac{\partial f_a}{\partial t}+\mathbf{v}_a\cdot\frac{\partial f_a}{\partial\mathbf{r}}+\frac{\mathbf{F}}{m}\cdot\frac{\partial f_a}{\partial\mathbf{v}_a}=(\frac{\delta f_a}{\delta t})_c$$

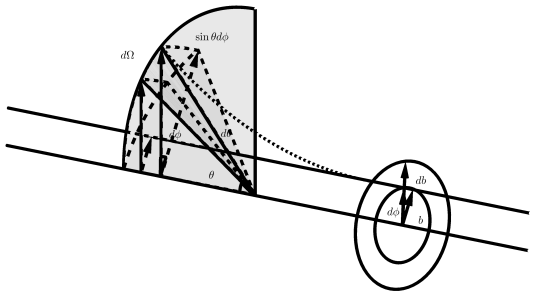

图 1: 微分散射截面

则有

$$\frac{\partial f_a}{\partial t} + \mathbf{v}_a \cdot \frac{\partial f_a}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial f_a}{\partial \mathbf{v}_a} = -\frac{\partial}{\partial \mathbf{v}_a} \cdot (f_a(\mathbf{r}, \mathbf{v}_a, t - \Delta t)\langle \Delta \mathbf{v}_a \rangle) + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v}_a \partial \mathbf{v}_a} : (f_a(\mathbf{r}, \mathbf{v}_a, t - \Delta t)\langle \Delta \mathbf{v}_a \Delta \mathbf{v}_a \rangle)$$

即为 Fokker-Planck 方程。

5. 由库仑碰撞确定转移概率 $\psi(\mathbf{v}_a - \Delta\mathbf{v}_a, \Delta\mathbf{v}_a)$ 的具体表达式，进而求出摩擦系数 $\langle \Delta \mathbf{v}_a \rangle$ 和扩散系数 $\langle \Delta \mathbf{v}_a \Delta \mathbf{v}_a \rangle$

(a). 一个服从 $f_a(\mathbf{v}_a)$ 分布的粒子与一个服从 $f_b(\mathbf{v}_b)$ 分布的粒子相碰时，a 粒子落入立体角元 $d\Omega$ 的概率即微分散射截面为

$$\frac{d\sigma}{d\Omega}$$

$$d\sigma = b d\phi db,\ d\Omega = \sin\theta d\theta d\phi \tag{8}$$

其中 $d\sigma$ 为散射截面的微元，b 为碰撞参数，θ 为散射角，ϕ 为方位角。$\frac{d\sigma}{d\Omega}$ 表示单位时间，散射到单位立体角 $d\Omega$，散射角范围在 θ 和 $\theta + \Delta\theta$ 之间的粒子数与单位时间内通过单位横截面 $d\sigma$ 的入射粒子数之比。

(b). 在单位时间内，一个服从 $f_a(\mathbf{v}_a)$ 分布的粒子与一群服从 $f_b(\mathbf{v}_b)$ 的粒子相碰时，$\mathbf{v}_a$ 和 $\mathbf{v}_b$ 表示 a 类粒子和 b 类粒子在质心系中的速度，a 类粒子落入立体角元 $d\Omega$ 的概率正比于

$$\frac{d\sigma}{d\Omega} \Delta t |\mathbf{v}_a - \mathbf{v}_b| f_b(\mathbf{v}_b) dv_b$$

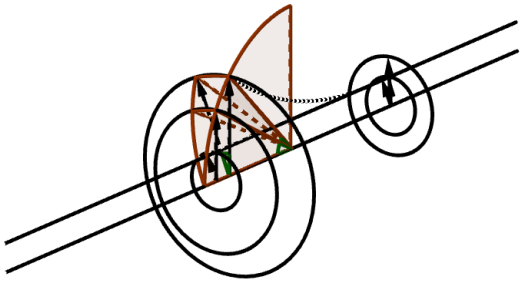

图 2: 微分散射截面

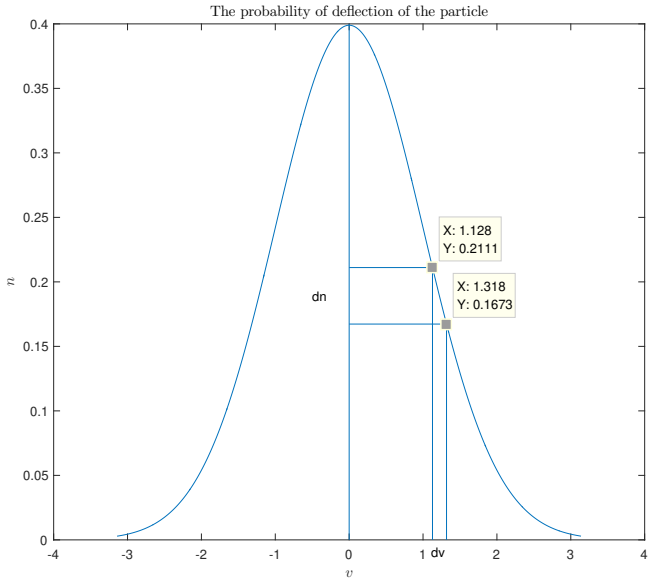

图 3: 粒子的速度分布

表示单位时间内速度为 $\mathbf{v}_a$ 的粒子和一群 $f_b(\mathbf{v}_b)$ 中速度为 $\mathbf{v}_b$，总粒子数为 $\Delta t|\mathbf{v}_a - \mathbf{v}_b| f_b(\mathbf{v}_b) d\mathbf{v}_b$ 的粒子碰撞后，落入立体角 $d\Omega$ 的概率。则有

$$\psi(\mathbf{v}_a, \Delta\mathbf{v}_a) d(\Delta\mathbf{v}_a) = \Delta t \sum_b |\mathbf{v}_a - \mathbf{v}_b| d\sigma f_b(\mathbf{v}_b) d\mathbf{v}_b \frac{d\sigma}{d\Omega} \tag{9}$$

带入公式 (5),(6) 得

$$\langle \Delta\mathbf{v}_a \rangle = \sum_b \int d\mathbf{v}_b f(\mathbf{v}_b) \int d\Omega |\mathbf{v}_a - \mathbf{v}_b| (\Delta\mathbf{v}_a) \frac{d\sigma}{d\Omega}$$

$$\langle \Delta\mathbf{v}_a \Delta\mathbf{v}_a \rangle = \sum_b \int d\mathbf{v}_b f(\mathbf{v}_b) \int d\Omega |\mathbf{v}_a - \mathbf{v}_b| (\Delta\mathbf{v}_a \Delta\mathbf{v}_a) \frac{d\sigma}{d\Omega}$$

由库仑散射公式

$$tan\frac{\theta}{2} = \frac{b_\perp}{b}$$

其中

$$b_\perp = \frac{z_a z_b e^2}{4\pi m_r \varepsilon_0 v^2}$$

$$\lambda_D = \frac{\mathbf{v}_{th}}{\omega_{pe}} = \frac{\mathbf{v}_{th}}{(\frac{ne^2}{m_e \varepsilon_0})^{1/2}}$$

其中 z_a, z_b 为入射粒子和靶粒子的核电荷数，m_r 为约化质量

$$m_r = \frac{m_a m_b}{m_a + m_b}$$

ε_0 为真空介电常数，$\mathbf{v}$ 为无穷远处 a，b 类粒子相对速度，$\mathbf{v} = \mathbf{v}_a - \mathbf{v}_b$。由

$$\frac{d\sigma}{d\Omega} = -\frac{b d\phi db}{\sin\theta d\theta d\phi} = -\frac{b}{\sin\theta}\frac{db}{d\theta}$$

其中的负号是因为 db 和 $d\theta$ 的符号相反，如下图

可得

$$\frac{d\sigma}{d\Omega} = \frac{b_\perp^2}{4\sin^4\frac{\theta}{2}} \tag{10}$$

此即为卢瑟福散射公式。

在质心坐标系中，入射粒子速度的改变量

$$\Delta\mathbf{v}_a = \frac{m_b}{m_a + m_b}\Delta\mathbf{v}$$

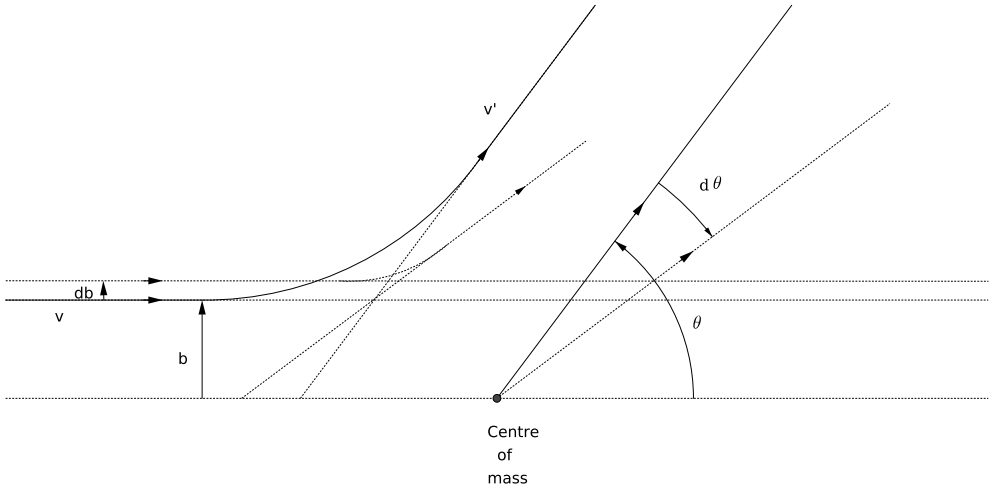

图 4: db 与 $d\theta$ 之间的关系

(c). 引入局部直角坐标系，以方便计算相对速度改变量 $\Delta\mathbf{v}$，局部坐标系的坐标基矢和实验室坐标系坐标基矢的关系为

$$\mathbf{e}_1' = \frac{\mathbf{v}}{v},\ \mathbf{e}_2' = \frac{\mathbf{e}_3 \times \mathbf{v}}{[(v^{(1)})^2 + (v^{(2)})^2]^{\frac{1}{2}}},\ \mathbf{e}_3' = \mathbf{e}_1' \times \mathbf{e}_2'$$

其中 $\mathbf{e}_1$，$\mathbf{e}_2$，$\mathbf{e}_3$ 指实验室坐标基矢（固定坐标系，实验室坐标系和质心坐标系坐标基矢相同），$\mathbf{e}_1'$，$\mathbf{e}_2'$，$\mathbf{e}_3'$ 指局部坐标基矢，$v^{(1)}$，$v^{(2)}$ 为相对速度 $\mathbf{v}$ 沿 $\mathbf{e}_1$，$\mathbf{e}_2$ 的分量。[2]

如下图

局部坐标系中粒子相对速度的改变如图

其中入射粒子初速度方向为 $\mathbf{e}_1'$，粒子偏转角 θ 和方位角 ϕ（图中角度值仅为参考）。由图可知，在局域坐标系中

$$\begin{aligned}\Delta\mathbf{v}_L &= \mathbf{v}' - \mathbf{v} \\ &= v[(\cos\theta - 1)\mathbf{e}_1' + \sin\theta\cos\phi\mathbf{e}_2' + \sin\theta\sin\phi\mathbf{e}_3'] \end{aligned} \tag{11}$$

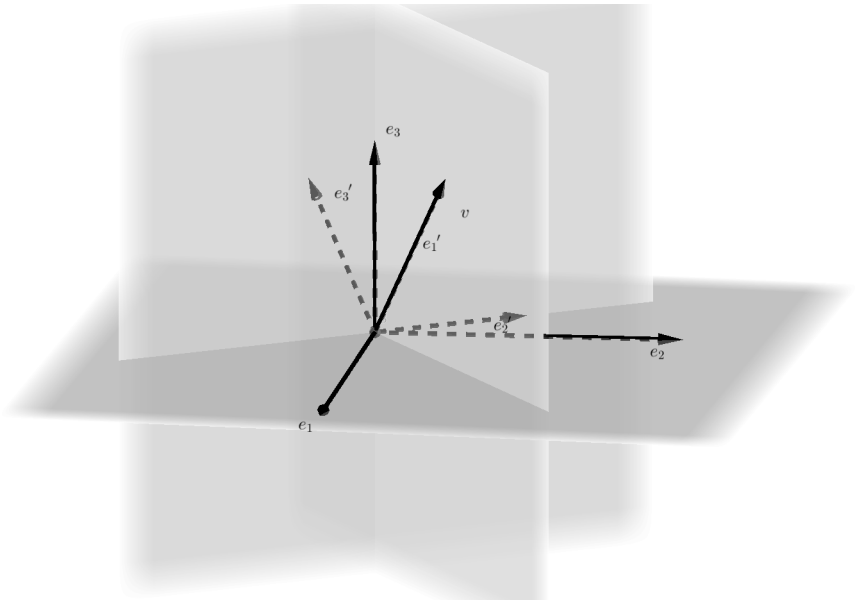

图 5: 局域坐标系和实验室坐标系之间的关系

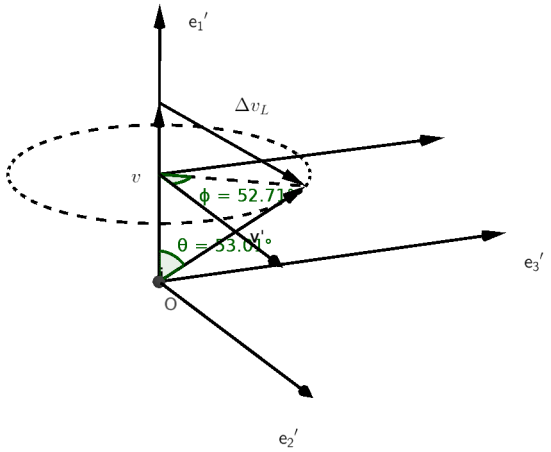

图 6: 局域坐标系中粒子相对速度的改变

$$\begin{aligned}\Delta\mathbf{v}_L\Delta\mathbf{v}_L =& v^2((\cos\theta-1)^2\mathbf{e}'_1\mathbf{e}'_1+(\cos\theta-1)\sin\theta\cos\phi\mathbf{e}'_1\mathbf{e}'_2+(\cos\theta-1)\sin\theta\sin\phi\mathbf{e}'_1\mathbf{e}'_3\\ &\sin\theta(\cos\theta-1)\cos\phi\mathbf{e}'_2\mathbf{e}'_1+\sin^2\theta\cos^2\phi\mathbf{e}'_2\mathbf{e}'_2+\sin^2\theta\cos\phi\sin\phi\mathbf{e}'_2\mathbf{e}'_3+\\ &\sin\theta(\cos\theta-1)\sin\phi\mathbf{e}'_3\mathbf{e}'_1+\sin^2\theta\sin\phi\cos\phi\mathbf{e}'_3\mathbf{e}'_2+\sin^2\theta\sin^2\phi\mathbf{e}'_3\mathbf{e}'_3)\end{aligned} \tag{12}$$

在实验室坐标系中有

$$\begin{aligned}\Delta v^{(\mu)} &= (\mathbf{e}_\mu\cdot\mathbf{e}'_\nu)\Delta v_L^{(\nu)}\\ \Delta v^{(\mu)}\Delta v^{(\nu)} &= (\mathbf{e}_\mu\cdot\mathbf{e}'_\sigma)(\mathbf{e}_\nu\cdot\mathbf{e}'_\omega)\Delta v_L^{(\sigma)}\Delta v_L^{(\omega)}\end{aligned}$$

其中上指标 μ, ν, σ, ω 代表在实验室坐标系下和局域坐标系下相对速度变化的分量，重复指标表示求和，第一个式子表明实验室坐标系下相对速度改变量的某分量的值为局域坐标系下相对速度改变量各个分量在实验系坐标系下该分量投影的和。

(d). 求解局域坐标系下立体角积分项，再将其转换到实验室坐标系下

令

$$\begin{aligned}\{\Delta v_L^{(\mu)}\} &= \int d\Omega(\Delta v_L^{(\mu)})v\frac{d\sigma}{d\Omega}\\ \{\Delta v^{(\mu)}\} &= \int d\Omega(\Delta v^{(\mu)})v\frac{d\sigma}{d\Omega}\end{aligned}$$

将 (10) 式，(11) 式代入得

$$\begin{aligned}\{\Delta v_L^{(1)}\} &= \int|\mathbf{v}|\frac{b_\perp^2}{4\sin^4\frac{\theta}{2}}\Delta v_L^{(1)}d\Omega\\ &= \int_0^\pi\int_0^{2\pi}\frac{b_\perp^2}{4\sin^4\frac{\theta}{2}}(\cos\theta-1)v^2\sin\theta d\theta d\phi\\ &= -2\pi v^2b_\perp^2\int_0^\pi\frac{2\sin^2\frac{\theta}{2}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{4\sin^4\frac{\theta}{2}}d\theta\\ &= -4\pi v^2b_\perp^2\int_0^\pi\frac{1}{\sin\frac{\theta}{2}}d\sin\frac{\theta}{2}\\ &= -4\pi v^2b_\perp^2[\ln 1-\ln(\sin 0)]\end{aligned}$$

对方程进行截断得 [2]

$$\begin{aligned}\{\Delta v_L^{(1)}\} &= -4\pi v^2 b_\perp^2 \ln(2/\theta_{min}) \\ &\approx -4\pi v^2 b_\perp^2 \ln\Lambda \\ &= -\frac{z_1^2 z_2^2 e^4}{4\pi\varepsilon_0^2 m_r^2 v^2}\ln\Lambda \end{aligned} \tag{13}$$

同理得

$$\{\Delta v_L^{(2)}\} = 0$$
$$\{\Delta v_L^{(3)}\} = 0$$

即有

$$\{\Delta \mathbf{v}_L\} = -\frac{z_1^2 z_2^2 e^4}{4\pi\varepsilon_0^2 m_r^2 v^2}\ln\Lambda \mathbf{e}_1'$$

$$\begin{aligned}\{\Delta\mathbf{v}_L\Delta\mathbf{v}_L\} =& \int v\frac{b_\perp^2}{4\sin^4\frac{\theta}{2}}\Delta\mathbf{v}_L\Delta\mathbf{v}_L d\Omega \\ =& \int_0^\pi\int_0^{2\pi} v^3\frac{b_\perp^2}{4\sin^4\frac{\theta}{2}}[(\cos\theta-1)^2\mathbf{e}_1'\mathbf{e}_1' \\ &+\sin^2\theta\cos^2\phi\mathbf{e}_2'\mathbf{e}_2' + \sin^2\theta\sin^2\phi\mathbf{e}_3'\mathbf{e}_3']\sin\theta d\phi d\theta \\ =& v^3 b_\perp^2(2\pi\int_0^\pi \sin\theta d\theta\mathbf{e}_1'\mathbf{e}_1' + \int_0^{2\pi}\cos^2\phi d\phi\int_0^\pi\frac{\sin^3\theta}{4\sin^4\frac{\theta}{2}}d\theta\mathbf{e}_2'\mathbf{e}_2' \\ &\int_0^{2\pi}\sin^2\phi d\phi\int_0^\pi\frac{\sin^3\theta}{4\sin^4\frac{\theta}{2}}d\theta\mathbf{e}_3'\mathbf{e}_3') \\ =& v^3 b_\perp^2[4\pi\mathbf{e}_1'\mathbf{e}_1' + \pi\int_0^\pi\frac{2\cos^3\frac{\theta}{2}}{\sin\frac{\theta}{2}}d\theta(\mathbf{e}_2'\mathbf{e}_2'+\mathbf{e}_3'\mathbf{e}_3')] \\ =& v^3 b_\perp^2[4\pi\mathbf{e}_1'\mathbf{e}_1' + 2\pi\int_0^\pi\frac{(1-\sin^2\frac{\theta}{2})\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}d\theta(\mathbf{e}_2'\mathbf{e}_2'+\mathbf{e}_3'\mathbf{e}_3')] \\ =& v^3 b_\perp^2[4\pi\mathbf{e}_1'\mathbf{e}_1' + (4\pi\int_0^\pi\frac{1}{\sin\frac{\theta}{2}}d\sin\frac{\theta}{2} - 4\pi\int_0^\pi\sin\frac{\theta}{2}d\sin\frac{\theta}{2})(\mathbf{e}_2'\mathbf{e}_2'+\mathbf{e}_3'\mathbf{e}_3')] \\ =& v^3 b_\perp^2\{4\pi\mathbf{e}_1'\mathbf{e}_1' + [4\pi\ln(\sin\frac{\theta}{2})|_0^\pi - 2\pi](\mathbf{e}_2'\mathbf{e}_2'+\mathbf{e}_3'\mathbf{e}_3')\}\end{aligned}$$

对方程进行截断得 [2]

$$\{\Delta\mathbf{v}_L\Delta\mathbf{v}_L\} = 4\pi v^3 b_\perp^2\mathbf{e}_1'\mathbf{e}_1' + v^3 b_\perp^2(4\pi\ln\Lambda - 2\pi)(\mathbf{e}_2'\mathbf{e}_2' + \mathbf{e}_3'\mathbf{e}_3') \tag{14}$$

进一步截断，大多数情况下 $\ln\Lambda \gg 1$，则有

$$\begin{aligned}\{\Delta\mathbf{v}_L\Delta\mathbf{v}_L\} &= 4\pi v^3 b_\perp^2 \ln\Lambda(\mathbf{e}_2'\mathbf{e}_2' + \mathbf{e}_3'\mathbf{e}_3') \\ &= \frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_r^2 v}\ln\Lambda(\mathbf{e}_2'\mathbf{e}_2' + \mathbf{e}_3'\mathbf{e}_3') \\ &= \frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_r^2 v}\ln\Lambda(\vec{\mathbf{I}} - \mathbf{e}_1'\mathbf{e}_1')\end{aligned}$$

由

$$\Delta v^{(\mu)} = (\mathbf{e}_\mu \cdot \mathbf{e}_\nu')\Delta v_L^{(\nu)}$$

$$\Delta v_a^{(\mu)} = \frac{m_b}{m_a + m_b}\Delta v^{(\mu)} = \frac{m_r}{m_a}\Delta v^{(\mu)}$$

得

$$\begin{aligned}\{\Delta v_a^{(\mu)}\} &= -\frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_r m_a v^2}\ln\Lambda(\mathbf{e}_\mu \cdot \mathbf{e}_1')\mathbf{e}_1' \\ &= -\frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_r m_a v^2}\ln\Lambda\frac{v^{(\mu)}}{v} \\ &= -\frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_r m_a}\ln\Lambda\frac{v^{(\mu)}}{v^3}\end{aligned}$$

则有

$$\{\Delta\mathbf{v}_a\} = -\frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_r m_a}\ln\Lambda\frac{\mathbf{v}}{v^3}$$

$$\begin{aligned}\langle\Delta\mathbf{v}_a\rangle &= \sum_b \int d\mathbf{v}_b f_b(\mathbf{v}_b)\{\Delta\mathbf{v}_a\} \\ &= -\frac{z_a^2 e^4}{4\pi\varepsilon_0^2 m_r m_a}\ln\Lambda\sum_b z_b^2\int d\mathbf{v}_b f_b(\mathbf{v}_b)\frac{\mathbf{v}}{v^3} \\ &= -\frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_r m_a} n_b \ln\Lambda\frac{\mathbf{v}}{v^3} \\ &= -\frac{z_a^2 z_b^2 \ln\Lambda}{4\pi n_b}\frac{n_b^2 e^4}{m_a^2\varepsilon_0^2}\frac{m_a}{m_r}\frac{\mathbf{v}}{v^3}\end{aligned}$$

其中 z_b 为常数，可以提出求和号外

$$n_b = \sum_b \int d\mathbf{v}_b f_b(\mathbf{v}_b)$$

由

$$\begin{aligned}\Delta v^{(\mu)}\Delta v^{(\nu)} &= (\mathbf{e}_\mu\cdot\mathbf{e}'_\sigma)(\mathbf{e}_\nu\cdot\mathbf{e}'_\omega)\Delta v_L^{(\sigma)}\Delta v_L^{(\omega)}\\ \Delta v_a^{(\mu)} &= \frac{m_b}{m_a+m_b}\Delta v^{(\mu)}\\ &= \frac{m_r}{m_a}\Delta v^{(\nu)}\end{aligned}$$

可得

$$\begin{aligned}\{\Delta v_a^{(\mu)}\Delta v_a^{(\nu)}\} =&(\frac{m_r}{m_a})^2(\mathbf{e}_\mu\cdot\mathbf{e}'_\sigma)(\mathbf{e}_\nu\cdot\mathbf{e}'_\omega)\Delta v_L^{(\sigma)}\Delta v_L^{(\omega)}\\ =&\frac{z_a^2z_b^2e^4}{4\pi\varepsilon_0^2m_a^2v}\ln\Lambda(\mathbf{e}_\mu\cdot\mathbf{e}'_\sigma)(\mathbf{e}_\nu\cdot\mathbf{e}'_\omega)(\vec{\vec{\mathbf{I}}}-\mathbf{e}'_1\mathbf{e}'_1):(\mathbf{e}'_\sigma\mathbf{e}'_\omega)\\ =&\frac{z_a^2z_b^2e^4}{4\pi\varepsilon_0^2m_a^2v}\ln\Lambda(\mathbf{e}_\mu\cdot\mathbf{e}'_\sigma)(\mathbf{e}_\nu\cdot\mathbf{e}'_\omega)(\delta_{\sigma\omega}-\delta_{1\sigma}\delta_{1\omega})\\ =&\frac{z_a^2z_b^2e^4}{4\pi\varepsilon_0^2m_a^2v}\ln\Lambda(\mathbf{e}_\mu\mathbf{e}_\nu):(\mathbf{e}'_\sigma\mathbf{e}'_\omega\delta_{\sigma\omega}-\mathbf{e}'_1\mathbf{e}'_1)\\ =&\frac{z_a^2z_b^2e^4}{4\pi\varepsilon_0^2m_a^2v}\ln\Lambda(\mathbf{e}_\mu\mathbf{e}_\nu):(\vec{\vec{\mathbf{I}}}-\mathbf{e}'_1\mathbf{e}'_1)\\ =&\frac{z_a^2z_b^2e^4}{4\pi\varepsilon_0^2m_a^2v}\ln\Lambda[\delta_{\mu\nu}-\frac{v^{(\mu)}v^{(\nu)}}{v^2}]\\ =&\frac{z_a^2z_b^2e^4}{4\pi\varepsilon_0^2m_a^2}\ln\Lambda\frac{1}{v}[\delta_{\mu\nu}-\frac{v^{(\mu)}v^{(\nu)}}{v^2}]\end{aligned}$$

则有

$$\{\Delta\mathbf{v}_a\Delta\mathbf{v}_a\} = \frac{z_a^2z_b^2e^4}{4\pi\varepsilon_0^2m_a^2}\ln\Lambda\frac{1}{v}(\vec{\vec{\mathbf{I}}}-\frac{\mathbf{vv}}{v^2})$$

$$\begin{aligned}\langle\Delta\mathbf{v}_a\Delta\mathbf{v}_a\rangle &= \sum_b\int d\mathbf{v}_b f_b(\mathbf{v}_b)\{\Delta\mathbf{v}_a\Delta\mathbf{v}_a\}\\ &= \frac{z_a^2e^4}{4\pi\varepsilon_0^2m_a^2}\ln\Lambda\sum_b z_b^2\int d\mathbf{v}_b f_b(\mathbf{v}_b)\frac{1}{v}(\vec{\vec{\mathbf{I}}}-\frac{\mathbf{vv}}{v^2})\\ &= \frac{z_a^2z_b^2\ln\Lambda}{4\pi n_b}\frac{n_b^2e^4}{m_a^2\varepsilon_0^2}\frac{1}{v}(\vec{\vec{\mathbf{I}}}-\frac{\mathbf{vv}}{v^2})\end{aligned}$$

由

$$v = [(v_a^{(\mu)}-v_b^{(\mu)})(v_a^{(\mu)}-v_b^{(\mu)})]^{\frac{1}{2}}$$

注意到

$$\begin{aligned}\frac{\partial}{\partial v_a^{(\mu)}}\frac{1}{v} &= \frac{\partial}{\partial v_a^{(\mu)}}\{\frac{1}{[(v_a^{(\mu)}-v_b^{(\mu)})(v_a^{(\mu)}-v_b^{(\mu)})]^{\frac{1}{2}}}\}\\ &= -\frac{1}{2}\frac{1}{[(v_a^{(\mu)}-v_b^{(\mu)})(v_a^{(\mu)}-v_b^{(\mu)})]^{\frac{3}{2}}}\frac{\partial}{\partial v_a^{(\mu)}}[(v_a^{(\mu)}-v_b^{(\mu)})(v_a^{(\mu)}-v_b^{(\mu)})]\\ &= -\frac{(v_a^{(\mu)}-v_b^{(\mu)})}{[(v_a^{(\mu)}-v_b^{(\mu)})(v_a^{(\mu)}-v_b^{(\mu)})]^{\frac{3}{2}}}\\ &= -\frac{v^{(\mu)}}{v^3}\end{aligned}$$

即

$$\frac{\partial}{\partial \mathbf{v}_a}\frac{1}{v} = -\frac{\mathbf{v}}{v^3}$$

$$\begin{aligned}\frac{\partial}{\partial v_a^{(\mu)}\partial v_a^{(\nu)}}v &= \frac{\partial}{\partial v_a^{(\nu)}}(\frac{\partial}{\partial v_a^{(\mu)}}v)\\ &= \frac{\partial}{\partial v_a^{(\nu)}}\{\frac{\partial}{\partial v_a^{(\mu)}}[(v_a^{(\mu)}-v_b^{(\mu)})(v_a^{(\mu)}-v_b^{(\mu)})]^{\frac{1}{2}}\}\\ &= \frac{\partial}{\partial v_a^{(\nu)}}\{\frac{1}{2}\frac{1}{[(v_a^{(\mu)}-v_b^{(\mu)})(v_a^{(\mu)}-v_b^{(\mu)})]^{\frac{1}{2}}}\frac{\partial}{\partial v_a^{(\mu)}}[(v_a^{(\mu)}-v_b^{(\mu)})(v_a^{(\mu)}-v_b^{(\mu)})]\}\\ &= \frac{\partial}{\partial v_a^{(\nu)}}\{\frac{v_a^{(\mu)}-v_b^{(\mu)}}{v}\}\\ &= \frac{\partial}{\partial v_a^{(\nu)}}(v_a^{(\mu)}-v_b^{(\mu)})\frac{1}{v}+\frac{\partial}{\partial v_a^{(\nu)}}\frac{1}{v}(v_a^{(\mu)}-v_b^{(\mu)})\\ &= \frac{\delta_{\mu\nu}}{v}-\frac{v^{(\mu)}v^{(\nu)}}{v^3} = \frac{1}{v}(\delta_{\mu\nu}-\frac{v^{(\mu)}v^{(\nu)}}{v^2})\end{aligned}$$

即

$$\frac{\partial}{\partial \mathbf{v}_a\partial \mathbf{v}_a}v = \frac{1}{v}(\vec{\mathbf{I}}-\frac{\mathbf{vv}}{v^2})$$

因此

$$\begin{aligned}\{\Delta\mathbf{v}_a\} &= \frac{z_a^2z_b^2e^4\ln\Lambda}{4\pi\varepsilon_0^2m_a^2}\frac{m_a}{m_r}\frac{\partial}{\partial\mathbf{v}_a}\frac{1}{v}\\ &= \Gamma_a\frac{m_a}{m_r}\frac{\partial}{\partial\mathbf{v}_a}\frac{1}{v}\end{aligned}$$

$$\begin{aligned}\{\Delta\mathbf{v}_a\Delta\mathbf{v}_a\} &= \frac{z_a^2z_b^2e^4}{4\pi\varepsilon_0^2m_a^2}\ln\Lambda\frac{\partial}{\partial\mathbf{v}_a\partial\mathbf{v}_a}v\\ &= \Gamma_a\frac{\partial}{\partial\mathbf{v}_a\partial\mathbf{v}_a}v\end{aligned}$$

其中

$$\Gamma_a = \frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_a^2} \ln\Lambda$$

将 $\{\Delta\mathbf{v}_a\}$, $\{\Delta\mathbf{v}_a\Delta\mathbf{v}_a\}$ 分别代入 $\langle\Delta\mathbf{v}_a\rangle$ 和 $\langle\Delta\mathbf{v}_a\Delta\mathbf{v}_a\rangle$ 中得

$$\begin{aligned}
\langle\Delta\mathbf{v}_a\rangle &= \sum_b \int d\mathbf{v}_b f_b(\mathbf{v}_b)\{\Delta\mathbf{v}_a\} \\
&= \sum_b \int d\mathbf{v}_b f_b(\mathbf{v}_b)\Gamma_a \frac{m_a}{m_r}\frac{\partial}{\partial\mathbf{v}_a}\frac{1}{v} \\
&= \Gamma_a \frac{\partial}{\partial\mathbf{v}_a}[\sum_b \frac{m_a}{m_r}\int d\mathbf{v}_b f_b(\mathbf{v}_b)\frac{1}{v}] \\
&= \Gamma_a \frac{\partial H_a}{\partial\mathbf{v}_a}
\end{aligned}$$

其中

$$H_a(\mathbf{v}_a) = \sum_b \frac{m_a}{m_r}\int d\mathbf{v}_b f_b(\mathbf{v}_b)\frac{1}{v} = \sum_b \frac{m_a+m_b}{m_b}\int d\mathbf{v}_b f_b(\mathbf{v}_b)\frac{1}{v}$$

同理

$$\begin{aligned}
\langle\Delta\mathbf{v}_a\Delta\mathbf{v}_a\rangle &= \sum_b \int d\mathbf{v}_b f_b(\mathbf{v}_b)\{\Delta\mathbf{v}_a\Delta\mathbf{v}_a\} \\
&= \sum_b \frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_a^2}\ln\Lambda \int d\mathbf{v}_b f_b(\mathbf{v}_b)\frac{\partial}{\partial\mathbf{v}_a\partial\mathbf{v}_a}v \\
&= \Gamma_a \frac{\partial^2}{\partial\mathbf{v}_a\partial\mathbf{v}_a}\sum_b \int d\mathbf{v}_b v f_b(\mathbf{v}_b) \\
&= \Gamma_a \frac{\partial^2 G_a}{\partial\mathbf{v}_a\partial\mathbf{v}_a}
\end{aligned}$$

其中

$$G_a = \sum_b \int d\mathbf{v}_b v f_b(\mathbf{v}_b)$$

因此有 Fokker-Planck 方程

$$\begin{aligned}
\frac{\partial f_a}{\partial t} + \mathbf{v}_a\cdot\frac{\partial f_a}{\partial\mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial f_a}{\partial\mathbf{v}_a} = &-\frac{\partial}{\partial\mathbf{v}_a}\cdot(f_a(\mathbf{r},\mathbf{v}_a,t-\Delta t)\Gamma_a\frac{\partial H_a}{\partial\mathbf{v}_a}) \\
&+\frac{1}{2}\frac{\partial^2}{\partial\mathbf{v}_a\partial\mathbf{v}_a}:(f_a(\mathbf{r},\mathbf{v}_a,t-\Delta t)\Gamma_a\frac{\partial G_a}{\partial\mathbf{v}_a\partial\mathbf{v}_a})
\end{aligned}$$

其中

$$H_a = \sum_b \frac{m_a + m_b}{m_b} \int d\mathbf{v}_b f_b(\mathbf{v}_b) \frac{1}{v}$$

$$G_a = \sum_b \int d\mathbf{v}_b v f_b(\mathbf{v}_b)$$

$$\Gamma_a = \frac{z_a^2 z_b^2 e^4}{4\pi\varepsilon_0^2 m_a^2} \ln\Lambda$$

参考文献

[1] M. N. Rosenbluth, W. M. Macdonald, and D. L. Judd, Phys. Rev., 107, 1 (1957)

[2] Cohen, Spitzer, and McRoutly, Phys. Rev. 80, 230 (1950)